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Coherence versus reliability of stochastic oscillators with delayed feedback

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For noisy self-sustained oscillators, both reliability, the stability of a response to a noisy driving, and coherence, understood in the sense of constancy of oscillation frequency, are important characteristics. Although both characteristics and techniques for controlling them have received great attention from researchers, owing to their importance for neurons, lasers, clocks, electric generators, etc., these characteristics were previously considered separately. In this paper, a strong quantitative relation between coherence and reliability is revealed for a limit cycle oscillator subject to a weak noisy driving and a linear delayed feedback, a convection control tool. The analytical findings are verified and enriched with a numerical simulation for the Van der Pol–Duffing oscillator.

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Recently, the robustness of response of a limit cycle oscillator to a noisy driving has attracted considerable attention from both experimentalists and theoreticians [1–10]. In different fields of science, related phenomena appear under different names. In neurophysiology the reliability of spiking neurons, which manifests itself as a coincidence of the responses of a single neuron to a repeated noisy input of a prerecorded wave form, attracts great attention [1]. In recent experiments with a noise-driven neodymium-doped yttrium aluminum garnet (Nd:YAG) laser [2], a similar property has been referred to as consistency. From the theoretical viewpoint, reliability and consistency are manifestations of the synchronization of uncoupled nonlinear oscillators receiving identical noisy driving [3–10].

Quantitatively, the stability of response, the reliability, is characterized by the largest Lyapunov exponent (LE). For smooth limit cycle oscillators the LE is negative [3,5,7], meaning that the system is reliable. However, a large noise may lead to a positive LE [3,6,7,11]; and antireliability for neuronlike systems in a "classic" experimental setup has even been forecast [9].

However, for some oscillatory systems not only is the response stability important, but also the coherence, i.e., the constancy of the oscillation frequency, which is measured by the diffusion constant of the oscillation phase. The coherence determines the precision of clocks (including biological ones [12]), the quality of electric generators, the susceptibility of an oscillatory system to external driving [13], and the predisposition to synchronization; a laser radiation should be coherent when one needs to focus the beam or redirect it without angular divergence, etc. In Ref. [13] (followed by the methodologically closely related Ref. [14]) an extremely efficient technique for controlling coherence by a weak delayed feedback has been proposed and theoretically analyzed (a successful experimental implementation of this technique for a laser in the chaotic regime has been reported in Ref. [15]). Remarkably, due to the time-shift symmetry, a noiseless limit cycle system is neutrally stable and remains such in the presence of delayed feedback. But in the presence of noise the delayed feedback utilized for controlling the coherence may considerably affect the response stability.

It is noteworthy that, in the presence of both delay and noise (or irregularities), the process is no longer Markovian; therefore, one may not apply such well-elaborated tools as the conventional Fokker-Planck equation, and *ad hoc* statistical methods are employed for studies [16–19]. This paper presents both analytical and numerical results on the reliability of noise-driven limit cycle oscillators subject to delayed feedback control, suggesting an effective means for controlling the reliability. Analysis of these results in the context of controlling coherence reveals strong quantitative relations [Eq. (13)] between the reliability and the coherence. The disclosed fact, that a high reliability occurs for a weak coherence, and vice versa the weaker the reliability the higher the coherence, imposes important limitations on implementation of this conventional control technique. Imperfect cases are also discussed.

In order to demonstrate numerically the relationship between coherence and reliability, a simulation for a noisy Van der Pol oscillator,

$$\ddot{x}-\mu(1-x^2)\dot{x}+x=k[\dot{x}(t-\tau)-\dot{x}(t)]+\varepsilon\xi(t),$$

$$\langle\xi(t_1)\xi(t_2)\rangle=2\delta(t_1-t_2),\quad\langle\xi\rangle=0, \tag{1}$$

has been performed. Here μ describes the closeness to the Hopf bifurcation point, k and τ are the feedback strength and delay time, respectively, ε is the noise amplitude, and $\xi(t)$ is the normalized white Gaussian noise. In the presence of noise the oscillation phase $\varphi=-\arctan(x/\dot{x})$ diffuses according to $\langle[\varphi(t)-\langle\varphi(t)\rangle]^2\rangle\propto Dt$. The diffusion constant D quantifies the coherence of oscillations.

Figure 1 shows the effects of a linear delayed feedback on the diffusion constant (DC) and the Lyapunov exponent measuring the exponential growth rate of perturbations in the system (1). It is noteworthy that not only are the LE and the DC crucially magnified or suppressed simultaneously when τ/T_0 (here T_0 is the oscillation period of the control-free noiseless system) is an integer or half integer, but even their ratio remains nearly constant as τ changes [see Fig. 1(b)].

Let us develop a phase description of the system. One can parametrize the states of a limit cycle system on the limit cycle by the oscillation phase φ uniformly growing in the course of temporal evolution. Such an oscillator subject to weak noise and feedback stays in the vicinity of this cycle, and its evolution may still be described within the framework of the conventional phase approximation [20]. Close to the bifurcation point, i.e., for $\mu\to0$, the Van der Pol oscillator

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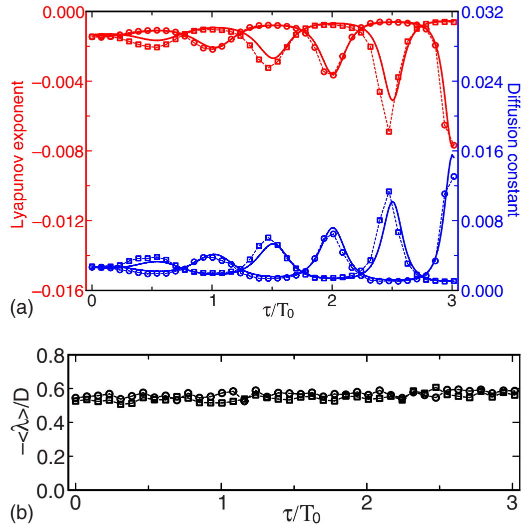

FIG. 1. (Color online) (a) Dependencies of Lyapunov exponent $\langle\lambda\rangle$ (upper graphs) and diffusion constant D (lower graphs) on delay time τ for the Van der Pol oscillator (1) with $\mu=0.7$ subject to white Gaussian noise of strength $\varepsilon^2=0.01$ and linear delayed feedback of strength $k=0.06$ (squares) and -0.06 (circles). Oscillation period of the control-free noiseless system $T_0\approx 2\pi/0.96$. The solid lines present the analytical dependencies [Eqs. (11) and (12)]. (b) The inconstancy of the ratio $-\langle\lambda\rangle/D$ is not resolvable against the background of the calculation inaccuracy.

has a nearly circular limit cycle, $x_0=2\cos\varphi$, $\dot{x}_0=-2\sin\varphi$, and the phase equation for the system (1) reads (cf. [13])

$$\dot{\varphi}=\Omega_0+ag[\varphi(t-\tau),\varphi(t)]+\varepsilon f[\varphi(t)]\circ\xi(t), \qquad (2)$$

where $\Omega_0=2\pi/T_0$ is the inherent cyclic frequency of the system, $a=k/2$, $g=\sin[\varphi(t-\tau)-\varphi(t)]$, the symbol "∘" indicates the Stratonovich form of the equation, and the 2π-periodic function $f(\varphi)$ is the sensitivity to noise. For an additive noise as in Eq. (1), $f(\varphi)=(2\Omega_0)^{-1}\cos\varphi$ (cf. [13]), but we keep f for generality. Note that, in Ref. [18], Eq. (2) was used to describe the evolution of the phase of an optical field in a laser with a weak optical feedback.

For a small perturbation α, one finds

$$\dot{\alpha}=a\cos[\varphi(t-\tau)-\varphi(t)][\alpha(t-\tau)-\alpha(t)]+\varepsilon f'[\varphi(t)]\alpha(t)\circ\xi(t)$$

(the prime stands for the derivative with respect to the argument). Therefore, the instant exponential growth rate $\lambda(t)$ obeys

$$\lambda(t)=a\cos[\varphi(t-\tau)-\varphi(t)]\left[\exp\left(-\int_{t-\tau}^{t}\lambda(t_1)dt_1\right)-1\right]+\varepsilon f'[\varphi(t)]\circ\xi(t); \qquad (3)$$

here we have made use of $\alpha(t)\propto\exp[\int^t\lambda(t_1)dt_1]$. Note that the LE is the mean value $\langle\lambda\rangle$.

For further analysis, it is more convenient to consider the equations in Itô form. Equations (2) and (3) read

$$\dot{\varphi}=\Omega_0+a\sin[\varphi(t-\tau)-\varphi(t)]+\varepsilon^2 f'f+\varepsilon f[\varphi(t)]\xi(t), \qquad (4)$$

$$\lambda(t)=a\cos[\varphi(t-\tau)-\varphi(t)]\left[\exp\left(-\int_{t-\tau}^{t}\lambda(t_1)dt_1\right)-1\right]+\varepsilon^2 f''f+\varepsilon f'[\varphi(t)]\xi(t). \qquad (5)$$

The terms ahead of the noisy ones describe the Stratonovich drift. Recall that, in Itô form (with the Stratonovich drift included explicitly), the instantaneous value $\varphi(t)$ is independent of the instantaneous value $\xi(t)$ taken at the same time moment t.

Let us explicitly introduce the mean frequency Ω and the instantaneous frequency deviation v; $\varphi\equiv\Omega t+\psi$, $\dot{\psi}=v$, $\langle v\rangle=0$. For a weak noise and a small feedback strength ($\varepsilon\ll 1, |a|\ll 1$) the instantaneous frequency fluctuations are small ($v\ll 1$), and Eqs. (4) and (5) yield, up to the main order of accuracy,

$$\Omega=\Omega_0-a\sin\Omega\tau, \qquad (6)$$

$$v=-a\cos\Omega\tau[\psi(t-\tau)-\psi(t)]+\varepsilon^2 f'f+\varepsilon f\xi(t), \qquad (7)$$

$$\lambda=a\{\cos\Omega\tau+\sin\Omega\tau[\psi(t-\tau)-\psi(t)]\}\times\left[\exp\left(-\int_{t-\tau}^{t}\lambda(t_1)dt_1\right)-1\right]+\varepsilon^2 f''f+\varepsilon f'\xi(t). \qquad (8)$$

Let us now find the LE. Assuming v and the fluctuating part $\tilde{\lambda}$ of λ, obeying

$$\tilde{\lambda}(t)\approx -a\cos\Omega\tau\int_{t-\tau}^{t}\tilde{\lambda}(t_1)dt_1+\varepsilon f'(\Omega t)\xi(t), \qquad (9)$$

to be Gaussian, one can employ the Furutsu-Novikov formula [21] to obtain from[1] Eq. (8)

$$\langle\lambda\rangle=a\cos\Omega\tau\left[-\langle\lambda\rangle\tau+1/2\left\langle\left(\int_{t-\tau}^{t}\tilde{\lambda}(t_1)dt_1\right)^2\right\rangle\right]-\varepsilon^2\langle f'^2\rangle_\varphi \qquad (10)$$

(here $\langle\cdots\rangle_\varphi$ stands for the average over the phase φ). The value $I\equiv\langle[\int_{t-\tau}^{t}\tilde{\lambda}(t_1)dt_1]^2\rangle$ can be evaluated from Eq. (9) [similarly to $\langle v(t_1)v(t_2)\rangle$ in Ref. [13]]:

$$I=\varepsilon^2\langle f'^2\rangle_\varphi\frac{\tau}{\pi}\int_{-\infty}^{+\infty}\left|\frac{ix}{1-e^{-ix}}+a\tau\cos\Omega\tau\right|^{-2}dx$$

$$=\varepsilon^2\langle f'^2\rangle_\varphi\begin{cases}\dfrac{2\tau}{1+a\tau\cos\Omega 2\tau} & \text{for } a\tau\cos\Omega\tau>-1,\\[2ex] \dfrac{2\tau}{\pi}\left(\dfrac{2}{1+a\tau\cos\Omega\tau}\right)^{8/7} & \text{for } a\tau\cos\Omega\tau<-1.\end{cases}$$

For $a\tau>1$, Eq. (6) exhibits multistability of the mean frequency Ω [13,17,18], which results in the violation of the

[1] Notice that Eq. (8) should not be linearized with respect to λ.

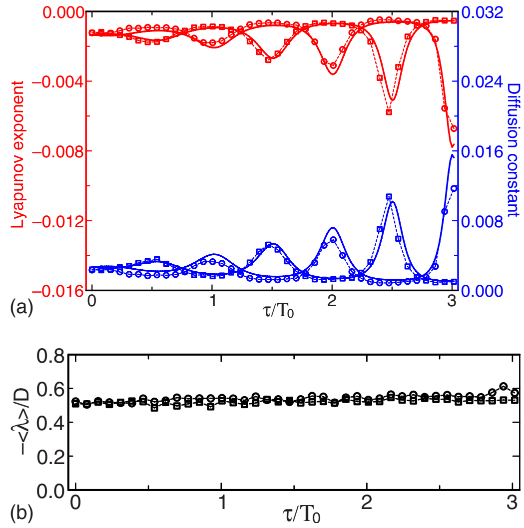

FIG. 2. (Color online) Same dependencies as in Fig. 1 for the same parameter values but for red Gaussian noise $\zeta(t) = T^{-1}\int_{t-T}^{t}\xi(t_1)dt_1$ with $T=1.2$. For notation see caption to Fig. 1.

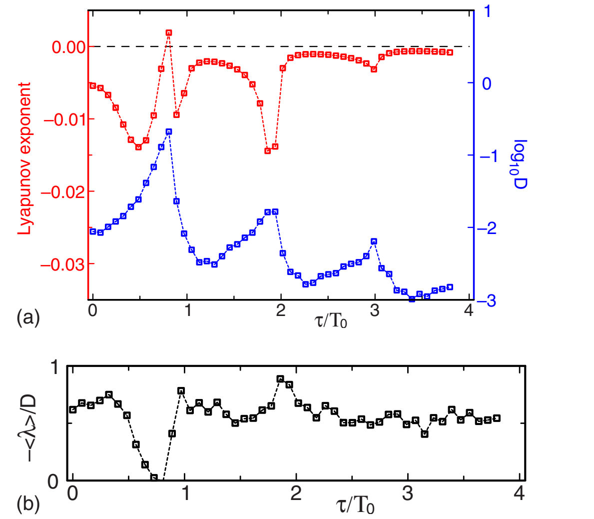

FIG. 3. (Color online) (a) Dependencies $\langle\lambda\rangle(\tau)$ (upper graph) and $D(\tau)$ (lower graph) for the Van der Pol–Duffing oscillator (14) with $\mu=0.2$ ($T_0\approx 2\pi/2.02$) subject to white Gaussian noise, $\varepsilon=0.05$, and the delayed feedback $k=0.06$. For description and notation of (b), see Fig. 1 caption.

basic assumptions of our analytical theory. Hence, the case $a\tau\cos\Omega\tau<-1$ may be ignored as meaningless, and after substitution of I Eq. (10) reads

$$\langle\lambda\rangle = -\frac{\varepsilon^2\langle f'^2\rangle_\varphi}{(1+a\tau\cos\Omega\tau)^2}, \tag{11}$$

whereas the DC has been already evaluated in Ref. [13]:

$$D = \frac{2\varepsilon^2\langle f^2\rangle_\varphi}{(1+a\tau\cos\Omega\tau)^2}. \tag{12}$$

Therefore,

$$-\frac{\langle\lambda\rangle}{D} = \frac{\langle f'^2\rangle_\varphi}{2\langle f^2\rangle_\varphi} = \text{const}, \tag{13}$$

which is 1/2 for $f(\varphi)=(2\Omega_0)^{-1}\cos\varphi$ as for the Van der Pol system in Fig. 1. Note that, due to the deformation of the limit cycle at $\mu=0.7$, relation (13) is more accurate than Eqs. (11) and (12) where the term $a\tau\cos\Omega\tau$ is specific to $g=\sin[\varphi(t-\tau)-\varphi(t)]$ (see Fig. 1 where $-\langle\lambda\rangle/D\approx 0.55$).

While deriving Eqs. (11) and (12), we nowhere utilized that the noise is δ correlated. Remarkably, the results remain valid for colored noise, e.g., red noise (see Fig. 2), the results for which coincide with those for white noise almost up to the numerical calculation inaccuracy, and $-\langle\lambda_{\rm red}\rangle/D_{\rm red}\approx 0.53$.

For a strong noise the phase description always leading to a negative LE is not applicable, and positive LEs have even been reported [3,6–9]. This case can be treated only numerically. For this reason, a simulation for the Van der Pol–Duffing oscillator

$$\ddot{x} - \mu(1-x^2)\dot{x} + x + x^3 = k[\dot{x}(t-\tau) - \dot{x}(t)] + \varepsilon\xi(t) \tag{14}$$

exhibiting positive LEs for a moderate noise [6] has been performed. Let us note that for a nonlarge noise (Fig. 3) the ratio $-\langle\lambda\rangle/D$ is changed by not more than 20%, while the DC and the LE are changed by a factor ≈ 20, in a broad range of τ (the only exception is in the interval $[T_0/2, T_0]$ near the domain where the LE is positive).

At $\varepsilon=0.374$ (Fig. 4), where the control-free Van der Pol–Duffing oscillator just becomes unstable (unreliable), $\langle\lambda\rangle|_{\tau=0}\approx 0$, the linear delayed feedback leads to maximal positive LEs for integer τ/T_0 and minimal (not always negative) LEs for half-integer τ/T_0. Concerning the interpretation of the dependence $\langle\lambda\rangle(\tau)$, note the following. In the absence of feedback control, intermittency of epochs of positive and negative local LEs ("local" means evaluated over a finite time interval) takes place and the transition to positive LE is related to a plain quantitative prevalence of the former over the latter (cf. [9]). The feedback affects (magnifies or suppresses) the local LEs over these epochs nonuniformly, thus shifting the balance between these epochs and bringing about a domination of positive local LEs for integer τ/T_0 and negative ones for half-integer τ/T_0. For positive "global" LEs, phase diffusion is owed mainly not to stochasticity but to chaos [samples of snapshot chaotic attractors (see Ref. [22]) are presented in Fig. 5]. As a result, here the DC is diminished where the LE is minimal.

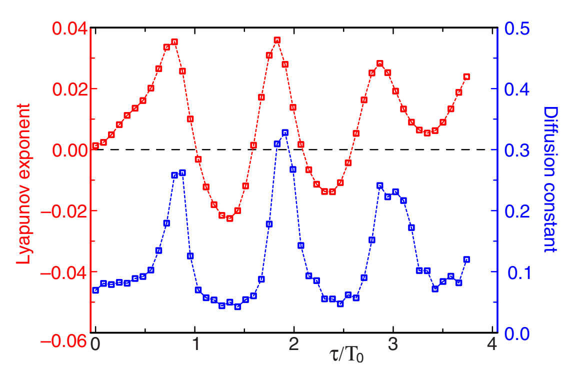

FIG. 4. (Color online) Same dependencies as in Fig. 3(a) for the same Van der Pol–Duffing system and feedback strength but a larger noise, $\varepsilon=0.374$.

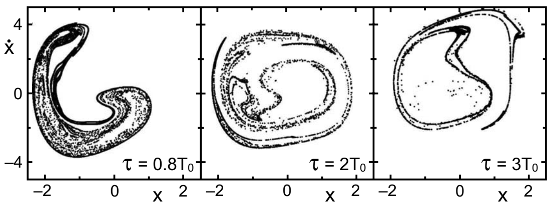

FIG. 5. Snapshots of the ensemble of 20 000 identical Van der Pol–Duffing oscillators (μ=0.2) driven by common white Gaussian noise (ε=0.374) with the fine structure for τ indicated in the plots (cf. to the LE in Fig. 4).

Summarizing, for a weak white or colored Gaussian noise,[2] a highly stable response (reliability) to a noisy driving is observed when phase diffusion is strong (i.e., the coherence is weak). Vice versa, for small diffusion (i.e., highly coherent oscillations) the response is weakly stable [Figs. 1 and 2, Eq. (13)]. In particular, this imposes strong limitations on the implementation of the technique of coherence improvement by virtue of a linear delayed feedback. For instance, in an ensemble of uncoupled identical self-sustained oscillators synchronized by a common external noisy driving, small intrinsic noise is always present and leads to spreading of oscillator phases: $\Delta\varphi \propto \varepsilon_{\rm in}/\sqrt{-\langle\lambda\rangle}$ ($\varepsilon_{\rm in}$ is the amplitude of intrinsic noise; cf. [6]). In such an ensemble the delayed feedback improvement of the coherence results in a mutual spreading of oscillator phases which may sometimes be undesirable.

For a strong noise capable of creating a positive Lyapunov exponent, i.e., antireliability, the chaotic contribution to phase diffusion may prevail over the stochastic one, and then enhanced coherence occurs for the maximal reliability (Fig. 4).

The detailed calculation of the Lyapunov exponent given above serves the purpose of disclosing the essentially different nature of various contributions to the Lyapunov exponent and the diffusion constant. However, the final quantitative effect of delayed feedback on these dissimilar properties of oscillatory systems somewhat surprisingly turns out to be identical. The reported phenomenon, valid for a general class of limit cycle oscillators, is thus neither intuitively expected nor trivial.

[2]Perhaps diffusionless noise such as blue noise should be excluded because in that case, unlike the case studied here, phase diffusion would be purely due to an interaction between the noise and the system nonlinearity.

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